INFERENCES ON POTENTIAL LIFETIMES OF PRODUCTS FROM DAMAGE DATA

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ABSTRACT. The damage on a specific quality characteristic of products over time is modeled by the Brownian motion model and geometric Brownian motion model, respectively. A discussion about the strength and shortcoming of these two stochastic processes is given. To easily evaluate the mean time to failure of test units from damage data, an algorithm is presented. An example is used to demonstrate the application of the proposed method. Simulation results show that the proposed method works well with a small bias and mean square error to infer the mean time to failure of test units.

Keywords: Accelerated Degradation Test, Maximum Likelihood Estimation Method, Mean Time To Failure, Percentile, Truncated Life Test.

1. Introduction. Many modern products are high-reliable and have high reliability due to the benefit from rapid advances of manufacturing technology. It becomes very difficult to gather lifetime information from high-reliable products through traditional life tests, such as the truncated life test, or censoring life test, in an affordable amount of time. Most often, only a few failed units can be found by the termination time of life testing, or even no any unit fails by the end of the life test. Hence, the accelerated degradation test (ADT) method would be adopted as an alternative.

The advantage of an ADT method is that engineers can assess the potential mean time to failure (MTTF) or percentiles of the lifetimes of test units from degradation data or damage data. Hence, statisticians and engineers paid more attention on the development of statistical inference procedures using the ADT method in the past few decades. A comprehensive summary for statistical modeling with ADT methods can be found in the referred articles of Park and Padgett (2006) and Lim and Yum (2011).

To implement an ADT experiment, engineers need to define an interested quality characteristic related with the lifetimes, then gather degradation measurements about the
specific quality characteristic over time. Most often, a diffusion effect exists among the degradation measurements if the test units are high-reliable. Hence, stochastic models such as the Brownian motion (BM) and geometric Brownian motion (GBM) are welcomed and often used to model the degradation data or damage data. Detailed discussions about the application of using a BM model or GBM model to infer potential lifetimes of the high-reliable products can be found in the referred articles such as Durham and Padgett (1997), Park and Padgett (2005), Lu (2004), Park and Padgett (2005), Onar and Padgett (2000) and Tsai et al. (2012).

This paper emphasizes on the discussion about the advantage and shortcoming of using a BM or GBM model to access the potential MTTF or percentiles of high reliable lifetimes. The modeling for BM and GBM is discussed in Section 2. In Section 3, the application of using a BM model and GBM model is demonstrated with an example. Some conclusions are given in Section 4.

2. Stochastic Models. Consider a continuous cumulative damage process, \( \{X_t, t = 1, 2, \ldots \} \) for lifetime inference. Assume that, as the stress level on a system is increased, the cumulative damage \( X_{n+1} \) after \( n+1 \) increments of stress is presented by

\[
c(X_{n+1}) = c(X_n) + D_n h(X_n),
\]

where \( D_n \) denotes damage incurred at the \((n+1)\)st increment, \( h(\cdot) \) is a damage model function, and \( c(\cdot) \) is a damage accumulation function. Model (1) can be expressed as

\[
dc(X_n) = h(X_n) dD_n
\]

Taking \( h(u) = 1 \) and \( c(u) = u \) in Model (1), the cumulative damage on a system at time \( t \) is labeled as \( X_t - X_0 = D_t \), where \( \{D_u : 0 \leq u \leq t\} \) is assumed as a BM process. It can be shown that \( D_t \) is normally distributed with mean \( vt \) and variance \( \beta^2 t \), where \( n \) is a positive drift parameter and \( \beta^2 \) is the diffusion parameter. Assume that the damage of a BM process increases from the starting level \( x_0 \), and ended at level \( x \) at time \( t \), labeled by \( x_t \). The probability density function (pdf) for \( X_t \) is given by (see Cox & Miller [10], Section 5.3)

\[
f(x; x_0, t) = \frac{1}{\beta \sqrt{t}} \phi \left( \frac{x-x_0-\nu t}{\beta \sqrt{t}} \right).
\]

Define the system failure time as the first passage time \( S = \inf \{u \mid X_u \geq C\} \), where \( C \) is the threshold. Then \( S \) has an inverse Gaussian distribution with the following pdf:

\[
gs(s; x_0, C) = \frac{\sqrt{\lambda}}{\sqrt{2\pi s^3}} \exp \left\{ -\frac{\lambda(s-\mu)^2}{2\mu^2 s} \right\}, s > 0,
\]

where \( \mu = (C-x_0) / \nu \), and \( \lambda = (C-x_0)^2 / \beta^2 \); See Chhikara & Folks [11]. To replace the parameters \( \mu \) and \( \lambda \) by \( \nu \) and \( \beta \), respectively, in (4), we obtain the following pdf,

\[
gs(s; x_0, C) = \frac{C-x_0}{\beta \sqrt{2\pi s^3}} \exp \left\{ -\frac{(C-x_0-\nu t)^2}{2\beta^2 s} \right\}, s > 0,
\]

for modeling the failed units in an ADT study.
The sample path of each survived unit over the interval \((0; t)\) is observed and measurements are gathered. Let \(A = \{X_i = x \mid x < C\}\). When \(C\) is large and the damage of a test unit is not cumulated over \(C\), the pdf of \(X\) condition on \(X < C\) in \((0, t)\) can be formulated as below:

\[
f_A(x; x_0, t) = \frac{1}{\beta \sqrt{t}} \phi \left( \frac{x - x_0 - vt}{\beta \sqrt{t}} \right) \left[ 1 - \exp \left\{ - \frac{2(C - x_0)(C - x)}{\beta^2 t} \right\} \right], x_0 < x < C \tag{6}
\]

The pitfall for a BM model is that the damage process is not increasing and can possibly be negative. Hence, the GBM model is used to replace the BM model to recover this pitfall. Taking \(h(u) = 1\) and \(c(u) = \log u\) the damage level of a test unit at time \(t\) can be presented as \(\log X_t - \log X_i = D_t\) \((7)\)

The stochastic process \(\{D_u : 0 \leq u \leq t\}\) has a BM model with drift coefficient \(\nu\) and diffusion \(\beta^2\). This type of damage model implies that the failure time is defined at which \(\log X_t\) reaches the critical value, \(\log C\). So the pdf of a failure time \(S\) for a GBM model is presented as:

\[
g_S(s; x_0, C) = \frac{1}{\sqrt{2\pi s}} \exp \left\{ - \frac{\lambda(s - \mu_t)^2}{2\mu_t^2 s} \right\} = \frac{C_0}{\beta \sqrt{2\pi s}} \exp \left\{ - \frac{(C_0 - \nu_s s)^2}{2\beta^2 s} \right\},
\]

where \(C_0 = \log C - \log x_0\) and \(\mu = C_0/\nu_L\), and \(\lambda = C_0^2/\beta^2\). If \(C\) is large and the damage of a test units is not cumulated over \(C\), the pdf of \(X\) condition on \(X < C\) in time interval \((0, t)\) is given as follows

\[
f_A(x; x_0, t) = \frac{1}{\beta x \sqrt{t}} \phi \left( \frac{\log x - \log x_0 - vt}{\beta \sqrt{t}} \right) \\
\times \left[ 1 - \exp \left\{ - \frac{2(\log C - \log x_0)(\log C - \log x)}{\beta^2 t} \right\} \right], x_0 < x < C \tag{9}
\]

The maximum likelihood estimation method is a common method to infer the MTTF or percentiles of the lifetimes of test units from the degradation data or damage data; See (Park and Padgett, 2006), (Park and Padgett, 2005), (Park and Padgett, 2005), and (Tsai et al. 2012). However, a hard effort in solving nonlinear equations is often needed to implement maximum likelihood estimation. If only the MTTF of test units is asked and no failure is found in the ADT, or the failure information is ignored. The following algorithm is recommended to infer the potential MTTF of test units. Estimation steps following the algorithm are easier to implement for engineers than using the maximum likelihood estimation method.

Algorithm. 

**Step 1:** Symbol \(y_{ij}\) as the \(j\)th degradation measurement of the \(i\)th test unit from a BM or GBM model. Gather degradation data, \((y_{ij}, t_j), j = 1, 2, \ldots, m, i = 1, 2, \ldots, n\), from an ADT experiment.
Step 2: Label $\bar{y}_j = \sum_{i=1}^{n} y_{ij}/n$ at time $t_j$, and let $z_j = \bar{y}_j - x_0$ for a BM model and $z_j = \log \bar{y}_j - \log x_0$ for a GBM model. Find the regression line, $z_j = vt_j$, $j = 1, 2, \ldots, m$.

Step 3: The MTTF of test units is evaluated as $(C - x_0)/v$ for a BM model or as $(\log C - \log x_0)/v$ for a GBM model.

![Damage paths of lumen degradation generated from the (a) BM and (b) GBM models.](image)

**Figure 1.** Damage paths of lumen degradation generated from the (a) BM and (b) GBM models.

3. An Example. Light emitting diode (LED) is one of the high-reliable products developed in the past few decades. Because the energy consumption of LEDs is low, compared with other competitors, LEDs have been successfully designed as an indicator for many devices in the lighting source market. However, the lifetime data set is difficult to collect due to the property of long life of LEDs. Most high-power LEDs have been claimed with a MTTF over 30,000 hours. ADT methods were then recommended to evaluate the potential lifetimes of LEDs. The degradation of lumen information can be gathered from an ADT for MTTF inference.

<table>
<thead>
<tr>
<th>Stochastic Model</th>
<th>Parameter</th>
<th>Absolute bias</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>BM</td>
<td>$\nu$</td>
<td>2.74E-07</td>
<td>9.57E-10</td>
</tr>
<tr>
<td></td>
<td>MTTF</td>
<td>0.00575</td>
<td>207.3702</td>
</tr>
<tr>
<td>GBM</td>
<td>$\nu$</td>
<td>4.53E-08</td>
<td>2.90E-11</td>
</tr>
<tr>
<td></td>
<td>MTTF</td>
<td>0.3512</td>
<td>1297.571</td>
</tr>
</tbody>
</table>

Table 1. Absolute biases and MSE of $\hat{n}$ and the estimated MTTF.

Figure 1(a) gives 20 damage paths of luminous, those were generated from a BM process with $\nu=0.008$, $\beta=0.015$ and $x_0=0$. Figure 1(b) gives 20 damage paths of luminous, those were generated from a GBM process with $\nu=0.002$, $\beta=0.0026$ and $x_0=1$. Usually, a high-power LED is defined as failed if its luminous loss reaches or over 30 percent. It follows that the MTTF = 30/0.008 = 3750 days for the damage data generated from the BM process, and the MTTF = $\log(1.3)/0.0002 = 1312$ days for the damage data generated from the GBM process.

The estimation process based on the proposed algorithm is repeated 5000 times, each for 20 test units ($n=20$). The absolute bias and mean square error (MSE) of estimates of $\nu$ and MTTF are evaluated, respectively. Table 1 gives the absolute biases and MSE of
estimates of $\nu$ and MTTF. We can find that all biases and MSE in table are small for the BM process or the GBM process. Based on the simulation results, we conclude that the estimation performance of the proposed method is satisfactory.

4. Final Remarks. In this paper, the damage of interested quality characteristic of high-reliable products is considered as a Brownian motion model or geometric Brownian motion model. An algorithm is provided to quickly access the potential mean time to failure of products from the degradation data or damage data. A simulation study based on the scenario of the luminous degradation process of light emitting diodes is conducted to demonstrate the application of the proposed method. However, the proposed method cannot provide interval inference information. That is, the estimation error cannot be evaluated through the proposed method.

In most ADT experiments of light emitting diodes, relative luminous degradation measurements are collected. So the geometric Brownian motion model is more reasonable for modeling such degradation process. The maximum likelihood estimation procedure with multiple stress variables, and the inference procedure on percentiles of product’s lifetimes from the degradation data or damage data can release more information about a lower confidence bound for percentiles in a shorter experimental time. These topics are interesting and will be investigated in the future.

REFERENCES