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ON A ROBUST COST-IMPORTANCE RATIO SHORTEST PATH PROBLEM

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ABSTRACT. *This paper considers a robust shortest path problem with the uncertain arc lengths and the importance of route. In order to present these uncertainties, box and ellipsoidal constraints based on the uncertainty set are introduced. The initial proposed model is not well-defined in the sense of mathematical programming, and robust programming approaches are introduced. Furthermore, deterministic equivalent transformations are performed in order to solve the problem analytically and explicitly.*

Keywords: Shortest Path Problem; Robust Programming; Importance of Route; Deterministic Equivalent Transformations

1. Introduction. Shortest path problem (SPP) is one of the most fundamental and important network optimization problems, and it is widely applied in many applications. Therefore, the shortest path problem has been studied extensively in many academic research fields such as computer science and operations research.

In previous studies of SPPs, arc lengths have been considered as deterministic values. These problems are efficiently and analytically solved using many algorithms developed by some outstanding researchers, and these algorithms are referred as the standard algorithms for SPPs in many existing studies. However, due to failure, maintenance or other reasons, different types of uncertainty are frequently encountered in practice. In these cases, probability theory has been used to deal with randomness, and many researchers have considered many studies on the stochastic SPP (Fam et al.,2005; Fu and Rillet, 1998; Thomas and White, 2007; Waller and Ziliaskopoulos, 2002). On the other hand, some researchers have considered that randomness is not the unique uncertainty in the real world and the probability distributions of arc lengths are sometimes difficult to acquire due to lack of historical data and reliable information. In this case, the arc lengths are approximately estimated by the expert, and fuzzy theory offers a powerful tool to deal with this case. Therefore, many researchers also have considered many studies on the fuzzy SPP (recently, (Chuang and Kung, 2006; Hernandes and Lamata, 2007; Iwamura and Shao, 2007; Keshavarz and Khorram, 2009)). In some studies, researches have proposed the generalized solution algorithm based on Dijkstra algorithm. In these studies, the weights of arcs are considered random variable, interval of fuzzy numbers, and analytical algorithm proposed by using different types of cost functions, partial order, possibility theory, and the other approaches.

Furthermore, some researchers considered to SPP in the case where the arc lengths are defined as uncertainty sets different from random distributions and fuzzy numbers. For instance, when a practical communication network is used to send packets from a source to a sink, it is often possible that an uncertain delay time occurs. Particularly, Karasan et al. (2001) and Bertsimas et al. (2003) modeled data uncertainty by assuming that each arc length belongs to an uncertainty set such as box and ellipsoidal constraints.

On the other hand, it is also important to consider how the trunk and backbone route in the network is developed. For instance, if a driver constructs a transportation route considering not only the total transportation time and but also the primary route with many gas-stations and wide roads, the driver need to determine the optimal route with both objects of minimizing the total transportation time and maximizing the importance of route. Thus, we need to set some value of importance to each arc length and to decide the shortest path considering the above-cited objects. As a solution approach, we may formulate a constrained SPP with the constraint to the degree of importance. However, this problem is generally NP-hard, and so it is hard to solve it strictly. Even if we deal with Lagrange multiplier method, we do not obtain a strict optimal path. In this paper, we propose a SPP both to maximize the total importance of route and to obtain the strict optimal path directly. As a similar situation of above-cited examples, some researchers considered the reliability for each edge and proposed the minimum cost-reliability ratio spanning tree problem (for instance, Ahuja (1988)). This problem is likely to arise in practice when cost as well as reliability is the criteria to be considered by the decision maker, and so it attempts to incorporate both the criteria into a single objective by yielding a path with low cost and high reliability. Therefore, using the idea of minimizing the total cost-reliability ratio, we propose a Cost-Importance Ratio Shortest Path Problem (CIRSPP) and develop the strict solution algorithm to the proposed model.

2. Mathematical definition.

A. Formulation of standard shortest path problem

In this section, we introduce the formulation of standard SPP. We define a path p_{ij} as a sequence of alternating node and arcs from i to j. If c_{ij} denotes a positive deterministic number associated with arc (i, j) corresponding to the cost necessary to traverse from i to j, the cost (arc length) of a path is the sum of the cost of the edge on the path. Therefore, we define an acyclic discrete network $G(N, A)$ consisting of a set of nodes $N=\{1,2,\ldots,n\}$ and m directed arcs $A \subseteq N \times N$, and formulate the SPP as the following problem:

$$\begin{aligned} &\text{Minimize } \sum_{(i,j)\in A} c_{ij}\, x_{ij} \\ &\text{subject to } \sum_{\{j\,|(i,j)\in A\}} x_{ij} - \sum_{\{j\,|(j,i)\in A\}} x_{ji} = \begin{cases} 1 & if\ i = s \\ 0 & if\ i \neq s,t \\ -1 & if\ i = t \end{cases} \quad (i = 1,2,\ldots,n) \qquad (1) \\ &x_{ij} \in \{0,1\}, \forall (i,j) \in A, s{:}\ starting\ node, t{:}\ terminal\ node \end{aligned}$$

In the case that all parameter are deterministic, we analytically and efficiently solve this shortest path problem using Dijkstra's algorithm.

However, costs or arc lengths of many real world applications are often not deterministic values. In these cases, it is quite appropriate to model the problem considering several uncertainties. Particularly, in the current world, many decision makers tend to avoid the worst case under several uncertain conditions. Therefore, in this paper, we introduce robust programming approaches for uncertain shortest path problem. Furthermore, we consider the importance of route, which is also uncertain due to the subjectivity of decision maker, and propose Cost-Importance Ratio Shortest Path Problem (CIRSPP).

B. Uncertainty sets

First, with respect to the uncertainty set $U_{\boldsymbol{c}}$ for arc length vector $\boldsymbol{c}$, we assume that $U_{\boldsymbol{c}}$ is a following ellipsoidal constraint:

$$U_{\boldsymbol{c}} = \{\boldsymbol{c}: \boldsymbol{c} = \boldsymbol{c}_n + Q_{\boldsymbol{c}}\boldsymbol{w}_{\boldsymbol{c}}, \|\boldsymbol{w}_{\boldsymbol{c}}\| \leq 1\} \tag{2}$$

where $\boldsymbol{c}_n = \left(c_{ij}^n\right)$ is the column vector of center values for $\boldsymbol{c}$ and $Q_{\boldsymbol{c}}$ is the positive definite matrix satisfying that all diagonal elements q_{ij}^c of $Q_{\boldsymbol{c}}$ are positive values and all ij-elements of $Q_{\boldsymbol{c}}$ are equal to be 0. Given $x_{ij}, \left((i,j) \in A\right)$, we may verify that the worst case length of shortest path $\sum_{(i,j)\in A} c_{ij}\, x_{ij}$ is attained when $\boldsymbol{w}_{\boldsymbol{c}} = Q_{\boldsymbol{c}}\boldsymbol{x}/\|Q_{\boldsymbol{c}}\boldsymbol{x}\|$. Therefore, the worst-case length is obtained as follows:

$$\sum_{(i,j)\in A} c_{ij}^n\, x_{ij} + \sqrt{\sum_{(i,j)\in A} q_{ij}^c\, x_{ij}^2} \tag{3}$$

In a similar way to $U_{\boldsymbol{c}}$, we define the uncertain set $U_{\boldsymbol{\alpha}}$ to the importance of each arc length $\alpha_{ij}, \left((i,j) \in A\right)$. We assume that $U_{\boldsymbol{\alpha}}$ is a following ellipsoidal constraint:

$$U_{\boldsymbol{\alpha}} = \{\boldsymbol{\alpha}: \boldsymbol{l} \leq \boldsymbol{\alpha} \leq \boldsymbol{u}\} \tag{4}$$

where $\boldsymbol{l}$ and $\boldsymbol{u}$ are column vectors of lower and upper values for $\boldsymbol{\alpha}$, respectively.

3. Proposition of Robust Cost-Importance Ratio Shortest Path Problem. In recent studies, some researchers proposed the minimum cost-reliability ratio spanning tree problem, whose numerator is the total cost and denominator is the total reliability of the spanning tree in the network. By minimizing the cost-reliability ratio, the decision maker obtain the optimal spanning tree considering maximizing the total reliability as well as minimizing the total cost. As a first step, in the case of deterministic parameters $\bar{c}_{ij}$ and $\bar{\alpha}_{ij}$, we propose a Cost-Importance Ratio Shortest Path Problem (CIRSPP) using the idea of minimizing the total cost-reliability ratio as follows:

$$\begin{aligned} &\text{Minimize } \frac{\sum_{(i,j)\in A} \bar{c}_{ij}\, x_{ij}}{\sum_{(i,j)\in A} \bar{\alpha}_{ij}\, x_{ij}} \\ &\text{subject to } \sum_{\{j\,|(i,j)\in A\}} x_{ij} - \sum_{\{j\,|(j,i)\in A\}} x_{ji} = \begin{cases} 1 & if\ i = s \\ 0 & if\ i \neq s,t \quad (i = 1,2,\dots,n) \\ -1 & if\ i = t \end{cases} \\ &x_{ij} \in \{0,1\}, \forall (i,j) \in A \end{aligned} \tag{5}$$

This deterministic problem is a fractional shortest path problem, and so it is possible to obtain the shortest path problem using the existing solution algorithm based on the Dinkelbach's algorithm (1967). However, in the propose approach, parameters $\hat{c}_{ij}$ and $\hat{\alpha}_{ij}$ are non-deterministic values represented as uncertainty sets. Therefore, in order to solve the proposed model analytically in terms of mathematical programming, we need to perform the deterministic equivalent transformations using robust programming. With respect to the objective function in the proposed model, minimizing the objective function is equivalently considered as the following form in terms of robust programming:

$$\text{Minimize } \frac{\sum_{(i,j)\in A} \hat{c}_{ij}\, x_{ij}}{\sum_{(i,j)\in A} \hat{\alpha}_{ij}\, x_{ij}} \rightarrow \text{Minimize } \max_{\hat{c}\in Q_{\boldsymbol{c}}, \hat{\alpha}\in U_{\boldsymbol{\alpha}}} \left\{\frac{\sum_{(i,j)\in A} \hat{c}_{ij}\, x_{ij}}{\sum_{(i,j)\in A} \hat{\alpha}_{ij}\, x_{ij}}\right\} \tag{6}$$

i.e.,

$$\text{Minimize} \max_{\hat{\boldsymbol{c}}\in Q_{\boldsymbol{c}}, \widehat{\boldsymbol{\alpha}}\in U_{\boldsymbol{\alpha}}} \left\{ \frac{\sum_{(i,j)\in A} \hat{c}_{ij}\, x_{ij}}{\sum_{(i,j)\in A} \widehat{\alpha}_{ij}\, x_{ij}} \right\} \rightarrow \text{Minimize} \frac{\max_{\hat{\boldsymbol{c}}\in Q_{\boldsymbol{c}}} \left\{\sum_{(i,j)\in A} \hat{c}_{ij}\, x_{ij}\right\}}{\min_{\widehat{\boldsymbol{\alpha}}\in U_{\boldsymbol{\alpha}}} \left\{\sum_{(i,j)\in A} \widehat{\alpha}_{ij}\, x_{ij}\right\}} \quad (7)$$

Therefore, the proposed model is equivalently transformed into the following problem considering the worst cases using formulae in Section 2-B:

$$\text{Minimize} \ \frac{\sum_{(i,j)\in A} c_{ij}^{n}\, x_{ij} + \sqrt{\sum_{(i,j)\in A} q_{ij}^{c}\, x_{ij}^{2}}}{\sum_{(i,j)\in A} l_{ij}\, x_{ij}}$$

$$\text{subject to} \ \sum_{\{j\,|(i,j)\in A\}} x_{ij} - \sum_{\{j\,|(j,i)\in A\}} x_{ji} = \begin{cases} 1 \ \ if\ i=s \\ 0 \ \ if\ i\neq s,t \ \ (i=1,2,\dots,n) \\ -1 \ \ if\ i=t \end{cases} \quad (8)$$

$$x_{ij} \in \{0,1\}, \forall (i,j)\in A$$

This problem is a fractional nonlinear SPP, and so it is also hard to solve this problem directly. In order to develop the strict solution algorithm, we consider the following auxiliary problem $\boldsymbol{P}(\lambda)$ introducing parameter λ:

$$\text{Minimize} \ \sum_{(i,j)\in A} c_{ij}^{n}\, x_{ij} + \sqrt{\sum_{(i,j)\in A} q_{ij}^{c}\, x_{ij}^{2}} - \lambda\left(\sum_{(i,j)\in A} l_{ij}\, x_{ij}\right)$$

$$\text{subject to} \ \sum_{\{j\,|(i,j)\in A\}} x_{ij} - \sum_{\{j\,|(j,i)\in A\}} x_{ji} = \begin{cases} 1 \ \ if\ i=s \\ 0 \ \ if\ i\neq s,t \ \ (i=1,2,\dots,n) \\ -1 \ \ if\ i=t \end{cases} \quad (9)$$

$$x_{ij} \in \{0,1\}, \forall (i,j)\in A$$

With respect to the relation between the main problem (8) and the auxiliary problem (9), the following theorem holds based on Dinkelbach's study.

Theorem 1 (Dinkelbach (1967)):

Let the objective function of $\boldsymbol{P}(\lambda)$ be denoted as $F(\lambda,\boldsymbol{x})$, and optimal solution of problem (9) be $\boldsymbol{x}^{*}$. Then, $\lambda^{*} = \sum_{(i,j)\in A} c_{ij}^{n}\, x_{ij}^{*} + \sqrt{\sum_{(i,j)\in A} q_{ij}^{c}\, \left(x_{ij}^{*}\right)^{2}} \Big/ \sum_{(i,j)\in A} l_{ij}\, x_{ij}^{*}$ if, and only if, $F(\lambda^{*},\boldsymbol{x}^{*})=0$.

Consequently, we develop the following strict solution algorithm for the proposed CIRSPP based on Dinkelbach's study.

Solution algorithm of main problem

STEP1: Set $\lambda_0 = 0$, and solve $\boldsymbol{P}(\lambda_0)$. If there is no feasible solution, then terminate this algorithm. In this case, there is no feasible solution of main problem and it is necessary to reset fuzzy goals. Else if there is an optimal shortest path, let the optimal shortest path of $\boldsymbol{P}(\lambda_0)$ be $\boldsymbol{x}_1$, and set $k=1$. Go to STEP2.

STEP2: Calculate $\lambda^{k} = \sum_{(i,j)\in A} c_{ij}^{n}\, x_{ij}^{k} + \sqrt{\sum_{(i,j)\in A} q_{ij}^{c}\, \left(x_{ij}^{k}\right)^{2}} \Big/ \sum_{(i,j)\in A} l_{ij}\, x_{ij}^{k}$, and go to STEP3.

STEP3: Solve $\boldsymbol{P}(\lambda_k)$, and denote the optimal solution by $\boldsymbol{x}_{k+1}$. Go to STEP4.

STEP4: With respect to sufficiently small number ε, if $F(\lambda_k,\boldsymbol{x}_k) < \varepsilon$,, then terminate this algorithm. In this case, $\boldsymbol{x}_{k+1}$. is the optimal shortest path. Else if $F(\lambda_k,\boldsymbol{x}_k) \geq \varepsilon$, $k \leftarrow k+1$ and return to STEP2.

In this solution algorithm, we need to repeatedly solve problem $\boldsymbol{P}(\lambda_k)$ on each λ^k, and so solving $\boldsymbol{P}(\lambda_k)$ is obviously the main task of this solution algorithm. With respect to $\boldsymbol{P}(\lambda_k)$, since each decision variable x_{ij} satisfies $x_{ij} \in \{0,1\}$, $x_{ij}^2 = x_{ij}$ holds. Therefore, problem (9) is equivalently transformed into the following problem in the case of λ^k:

$$\text{Minimize} \quad \sum_{(i,j)\in A} \left(c_{ij}^n - \lambda_k l_{ij}\right) x_{ij} + \sqrt{\sum_{(i,j)\in A} q_{ij}^c\, x_{ij}}$$

$$\text{subject to} \quad \sum_{\{j\,|(i,j)\in A\}} x_{ij} - \sum_{\{j\,|(j,i)\in A\}} x_{ji} = \begin{cases} 1 \ \ if\ i = s \\ 0 \ \ if\ i \neq s,t \ \ (i = 1,2,\dots,n) \\ -1 \ \ if\ i = t \end{cases} \tag{10}$$

$$x_{ij} \in \{0,1\}, \forall (i,j) \in A$$

With respect to the same type of problem (10), Geetha and Nair [8] proposed the bicriteria solution algorithm to stochastic minimum spanning tree problems. This algorithm is based on the study of Aneja and Nair (1979), which referred to the application of this algorithm to the other network problems. Furthermore, in terms of efficiency, the complexity of Geetha and Nair's algorithm is lower than other solution approaches. Therefore, based on Geetha and Nair's algorithm, we develop the strict solution algorithm for problem $\boldsymbol{P}(\lambda_k)$.

4. Conclusion. In this paper, we have considered a robust shortest path problem with the uncertain arc lengths and the total importance of route. In order to present uncertainties of arc length and importance, we have introduced ellipsoidal and box constraints based on the uncertainty set, and proposed the robust Cost-Importance Ratio Shortest Path Problem. Since the initial proposed model was not well-defined in the sense of mathematical programming, and so we have introduced robust programming approaches and performed deterministic equivalent transformations. Consequently, we have developed the strict solution algorithm based on some useful and efficient solution algorithms.

As a future study, we will improve the proposed solution algorithm more efficiently. Furthermore, we will study experimental studies using the proposed solution approach.

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