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INTERVAL ESTIMATION FOR THE QUANTILE OF A TWO-PARAMETER EXPONENTIAL DISTRIBUTION

JUAN WANG AND XINMIN LI
School of Sciences, Shandong University of Technology
Zibo, Shandong 255049, China
xmli@sdut.edu.cn

ABSTRACT. *In this paper, we consider the interval estimation for the quantiles of two-parameter exponential distributions. Based on bootstrapping and fiducial inferences, two methods for the interval estimation of quantiles are proposed. To evaluate the coverage probabilities and expected lengths of the two methods, a simulation study is conducted. The results indicate that the fiducial inference method performs well under all of the examined conditions.*

Keywords: Exponential Distribution; Quantile; Bootstrap; Fiducial Distribution; Coverage Probability

1. Introduction. Two-parameter exponential distribution, which occupies an important position in probability and statistical areas, has been widely used in practice, especially in the area of reliability. During the past few decades, there are many authors considered the statistical inferences for the two-parameter exponential distribution, see Nelson (1982), Lawless (1982), Bain and Engelhardt (1991), Balakrishnan and Basu (1995), and Meeker and Escobar (1998). It is well known that the quantiles of a random sample are the common used indicators to assess the reliability in statistical analysis. However, as the quantiles do not depend on nuisance parameters, there is no exact frequency method for interval estimation of quantiles. Consequently, there has not been much attempt to the study of the inferences on quantiles in recent literatures.

In some complicated situations, Efron (1979) proposed the bootstrap method for statistical inference. Free of population distributions and parameters, the bootstrap theory has been greatly developed and expanded for the last three decades, and now this technique is wildly used in various fields of statistics, see Hall (1988), DiCiccio and Efron (1996), Chen and Tong (2003), etc. Recently, the fiducial inference has attracted a great amount of attention due to its advantage of handling the inference problems under certain complex situations. Fisher proposed and discussed the fiducial inference firstly in 1935. David and Stone (1998) derived a generalized method to conduct the fiducial inference based on the function model. More recently, Li et al. (2005) and Hannig et al. (2006) further discussed the fiducial theory and developed a general method to construct the fiducial intervals.

The main work of this paper is to give the interval estimations for quantiles of two-parameter exponential distribution based on the bootstrap method and the fiducial method. Numerical simulations are conducted to compare the two methods mentioned above.

2. Interval Estimation of the Quantiles.

2.1. The Two-Parameter Exponential Distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample from the two-parameter exponential distribution with its probability density function (pdf) given by

$$p\left(x;\mu,\sigma\right)=\begin{cases}\frac{1}{\sigma}\exp\{-\frac{x-\mu}{\sigma}\}, & x>\mu\\ 0 & \text{, else}\end{cases}$$

where $\mu>0, \sigma>0$. By $F(x)=P(X\le x_p)=p$, the pth quantile x_p can be expressed as $x_p=u-\sigma\log(1-p)$ for any $p\in(0,1)$.

Assume the observations are the type II censored data $X_{(1)}\le X_{(2)}\le\cdots\le X_{(r)}$. It is obvious that $W=X_{(1)}$ and $V=\sum_{i=1}^{r}X_{(i)}+(n-r)X_{(r)}-nX_{(1)}$ are the complete sufficient statistics of the distribution and independently distributed. Then, the uniformly minimum-variance unbiased estimators (UMVUE) for μ and σ are

$$\hat{\mu}=X_{(1)}-\frac{1}{n(r-1)}\{\sum_{i=1}^{r}X_{(i)}+(n-r)X_{(r)}-nX_{(1)}\},$$

$$\hat{\sigma}=\frac{1}{r-1}\{\sum_{i=1}^{r}X_{(i)}+(n-r)X_{(r)}-nX_{(1)}\}.$$

Hence, the UMVUE of the pth quantile x_p is

$$\hat{x}_p=X_{(1)}-\frac{1}{n(r-1)}\{\sum_{i=1}^{r}X_{(i)}+(n-r)X_{(r)}-nX_{(1)}\}$$
$$-\frac{1}{r-1}\{\sum_{i=1}^{r}X_{(i)}+(n-r)X_{(r)}-nX_{(1)}\}\log(1-p).$$

2.2. The Bootstrap Method. The bootstrap method, proposed by Efron (1979), is often used to construct the confidence intervals for parameters. The main thought of the bootstrapping is to adopt the empirical probability distribution as a replacement of the unknown distribution of underlying population from which the original samples are drawn, and then construct new random variables basing on the independently distributed samples generated from empirical distribution, which is a substitution of the original samples, for further statistical inference. The bootstrap procedure for the calculation of confidence interval of the pth quantile is given as follows:

(1) Calculate the empirical distribution function based on the type II censored data $X_{(1)}\le X_{(2)}\le\cdots\le X_{(r)}$.

(2) By Monte Carlo simulation method, generate a sample $x_1^*\le x_2^*\le\cdots\le x_r^*$ from the empirical distribution with size r randomly.

(3) Based on the random sample, calculate the UMVUE of x_p by $\hat{x}_p=\hat{\mu}-\hat{\sigma}\log(1-p)$.

(4) Repeat step (2)-(3) for N (=5000) times and get N corresponding $\hat{x}_p$.

(5) For the given confidence coefficient $1-\alpha$, sort the $\hat{x}_p$'s in an ascending order, that is, $\hat{x}_{p(1)} \le \hat{x}_{p(2)} \le \cdots \le \hat{x}_{p(N)}$.

Find their $\alpha/2$ and $1-\alpha/2$ percentiles denoted by $\hat{x}_{p,L}$ and $\hat{x}_{p,U}$, respectively. Then the bootstrap confidence interval of the quantile x_p is $[\hat{x}_{p,L}, \hat{x}_{p,U}]$.

2.3. The Fiducial Method. Let $P_\eta(\cdot)$ denote the pdf of the random variable X with its sample space X, where η is the unknown parameter in the parameter space Ω. $\theta = \theta(\eta)$ is a real-valued parameter function of interest.

Definition 2.1. Suppose that there exist a random variable E with known distribution on space Ξ and a function $h(\eta, e)$ from $\Omega \times \Xi$ to X such that $X = h(\eta, E)$ for all $\eta \in \Omega$. Furthermore, if for any observation $x \in \mathrm{X}$ and $e \in \Xi$, the equation $x = h(\eta, e)$ has a unique solution in Ω, denoted by $\eta_x(e)$, then the distribution of $\theta(\eta_x(E))$ is called the fiducial distribution of $\theta = \theta(\eta)$.

In the following, we will give the confidence interval of the quantile x_p using the fiducial method. Noted that

$$W = X_{(1)},\ V = \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)} - nX_{(1)}$$

are complete sufficient statistics, and independently distributed. By

$$X = \frac{2n(W-\mu)}{\sigma} \sim \chi^2(2),\ Y = \frac{2V}{\sigma} \sim \chi^2(2r-2)$$

we have

$$(W, V) = (\frac{\sigma X}{2n} + \mu, \frac{\sigma Y}{2})$$

For a given observation (w, v) and (x, y), the equation

$$(w, v) = (\frac{\sigma x}{2n} + \mu, \frac{\sigma y}{2})$$

has a unique solution

$$(\mu, \sigma) = (w - \frac{vx}{ny}, \frac{2v}{y}).$$

Consequently, the fiducial distribution of the pth quantile is

$$F_x(x_p) = P_r(w - \frac{vX}{nY} - \frac{2v}{Y}\log(1-p) < x_p)$$

Hence, for given $\alpha \in (0,1)$, the fiducial confidence interval of x_p is $[x_p(\alpha/2), x_p(1-\alpha/2)]$, where $x_p(\gamma)$ is the 100γ quantile of x_p.

Generally speaking, there exists no explicit expression for the fiducial distribution of x_p, and it is difficult to find a numerical solution. However, the simulation method would be

helpful to conduct the calculation of the fiducial intervals.

(1) For given data, set the size of the simulated samples N large enough, say $N = 5000$.

(2) For $i = 1,2,\ldots,N$, generate X_i and Y_i from $\chi^2(2)$ and $\chi^2(2(r-1))$, respectively.

(3) Compute $$x_{p,i} = w - \frac{vX_i}{nY_i} - \frac{2v}{Y_i}\log(1-p).$$

(End N loop)

Denote $x_p(\gamma)$ as the 100γ percentile of $\{x_{p,1}, x_{p,2}, \cdots, x_{p,N}\}$. Then the fiducial confidence interval of x_p is $[x_p(\alpha/2), x_p(1-\alpha/2)]$.

3. Simulation Results. This section is devoted to the comparison of the bootstrap method and fiducial method using numerical simulation. In general, the mutual comparison of the above two methods should take into account the following properties: the coverage probabilities (CP) and the expected lengths (EL) of the intervals. The inference procedures with larger CP are desired firstly, and then a shorter EL would be considered as the indication of more accurate interval estimation.

In order to evaluate the interval estimation of the above two methods, we here apply Monte Carlo simulation to estimate CP and EL. For given μ and σ, generate M samples, compute their bootstrap intervals and fiducial intervals under the nominal level $1-\alpha$ using the related algorithms put forward in section 2, and finally calculate the proportion of the M intervals containing x_p and the average interval lengths. In the simulation procedure, we set $\mu = 2$, $\sigma = 4$, the sample size $n = 12$, the number of observed censored data $r = 8$, the confidence coefficient $1-\alpha = 95\%$ and $M = 3000$. The simulation results are shown in Table 1.

TABLE 1. The simulated CP and EL

P	Bootstrap intervals		Fiducial intervals	
	CP	EL	CP	EL
0.15	0.7723	1.3698	0.9427	1.8392
0.25	0.8833	1.8014	0.9407	2.3308
0.35	0.9217	2.5106	0.9477	3.2295
0.45	0.9300	3.4505	0.9533	4.3433
0.50	0.9410	3.9647	0.9553	5.1475
0.55	0.9447	4.6237	0.9513	5.7954
0.65	0.9457	6.1248	0.9480	7.4689
0.75	0.9477	8.2573	0.9527	10.0852
0.85	0.9447	11.3563	0.9547	14.9056
0.95	0.9583	18.5748	0.9557	24.8511

The numerical results in Table 1 indicate that the CPs of the fiducial intervals are close to $1-\alpha$, and apparently larger than that of the bootstrap intervals for small p. Under this condition, the fiducial method performs more satisfactorily than the bootstrap method. When the value of *p* is moderately large, the CPs of the two kinds of intervals are close to each other. When it comes to the ELs of the confidence intervals, nevertheless, the bootstrap method performs better. Therefore, we can conclude that the fiducial method would not be affected by the value of *p*, and the bootstrap method could be well accepted only with moderate to large values of *p*.

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